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Statistics of Atmospheric Gust Patterns Expressed in Terms of Energy and Entropy

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This paper reviews and extends an approach to the gust-load prediction problem, in which the probabilistic model of turbulence is expressed in the time plane in terms of a spectral energy function, dependent on the power spectral density, and the associated aircraft load is calculated as the response to a worst-case input that causes maximum response subject to a constraint on the spectral energy. It is described how the existing specification of the airworthiness requirements, which are based on the assumption that turbulence can be modeled as an ensemble of Gaussian patches, can be cast rigorously in this form. When the input is non-Gaussian, however, as is the case with measured velocity components in severe turbulence, the simple relationship between the power spectral density and probability no longer holds. In this instance, it is shown how the probability of a turbulence fluctuation may be specified in terms of a generalized energy function, which combines the spectral energy with the information entropy, which is a measure of signal complexity. The associated aircraft design load may then be calculated as the response to the worst-case input that causes maximum response subject to a constraint on the generalized energy. Applications to both linear and nonlinear aircraft dynamic responses are described.

Nomenclature

$\bar{A}$	=	measure of system response found by power spectral density analysis
a	=	empirical parameter
b	=	empirical parameter
C	=	normalization constant
E	=	expected value
$e(n)$	=	value of $e(n;q)$ for $q \approx 4$, representing tails of distributions
$e(n;q)$	=	energy-reduction factor
$e(S_I)$	=	entropy-based energy-reduction factor
$\bar{F}$	=	mother wavelet
$f(X)$	=	probability density function
g	=	wavelet representation
H	=	gust gradient distance
$H(i\omega)$	=	frequency-response function
H_n	=	shape of the gust pattern comprising n wavelets
$\{i\}$	=	states of system
k	=	gust scaling index
L	=	scale length
m_i	=	wavelet amplitude
n	=	number of elementary components in the gust or wavelet pattern
$p(n;q)$	=	pattern amplitude factor
$p_G(n)$	=	value of $p(n;q)$ in the Gaussian process
$\{p_i\}$	=	probability distribution over states $\{i\}$
q	=	moment-order parameter
$r(n;q)$	=	amplitude-reduction factor
S_I	=	information entropy
s_i	=	wavelet location (in space or time)
T	=	length of the time interval
T_i	=	wavelet scale
t	=	time
U	=	generalized energy function
U_σ	=	turbulence intensity associated with design load
u_k	=	gust intensity parameter
V	=	aircraft true airspeed
W	=	probability functional
w	=	turbulence-velocity increment
$w(\xi)$	=	probability density of ξ
X	=	fluctuation amplitude
X_q	=	fluctuation amplitude associated with q
$y(H_n, U)$	=	magnitude of response
y_d	=	design load
z	=	function used in the definition of a probability functional
$\Xi(i\omega)$	=	Fourier transform of $\xi(t)$
ξ	=	random variable
$\xi(t)$	=	random function, filter input
ϕ	=	power spectral density
$\Psi(i\omega)$	=	Fourier transform of $\psi(t)$
$\psi(t)$	=	filter output
ω	=	frequency

I. Introduction

THE design-load requirements in the airworthiness regulations for flight in turbulence [1] are based on a probabilistic model of the turbulence, together with a specified method for calculating the related loads in the frequency plane, applicable when the aircraft dynamic response is linear. The model of turbulence assumes that, at least in patches of limited extent, components of turbulence can be represented as samples of a Gaussian process having a prescribed power spectral density (PSD). As will be described, it has been shown that, rather than employing the method of load calculation specified in the regulations, which is implemented in the frequency plane, the loads may equivalently be calculated in the time plane. In this implementation, the turbulence model is expressed in terms of an energy function that is dependent on the PSD, which we shall refer to as a *spectral* energy function. The associated loads are calculated by a variational method as the response to a worst-case input that causes maximum response subject to a constraint on the spectral energy.

Subsequent sections of this paper will be concerned with generalizations of this variational method to non-Gaussian turbulence models and to aircraft nonlinear response. Starting from the central role played by spectral energy in the Gaussian model, a conceptual framework for the non-Gaussian model is introduced that combines spectral energy with information entropy, which is a measure of the complexity of a turbulence fluctuation. Applications

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will be described to both linear and nonlinear aircraft dynamic responses.

In a review of the required mathematical foundations, to be presented in Sec. II, the probabilistic model of a fluctuating signal is first expressed in the time plane in terms of a generalized energy function that is applicable to both Gaussian and non-Gaussian processes and that reduces to spectral energy in the Gaussian case. It is shown how the associated response statistics of a dynamical system may then be calculated in terms of the worst-case response to a deterministic input subject to a constraint on its generalized energy. Not only can the design-load requirements for flight in turbulence [1], which incorporate the assumption that the aircraft dynamic response be linear, be cast rigorously in this form, but the formulation in terms of generalized energy and worst-case analysis can be extended to treat the problem of nonlinear response in a consistent manner.

II. Mathematical Foundations: Probability and Energy

To specify a probability distribution on a class of fluctuating signals requires us, in principle, to be able to integrate in an infinite-dimensional function space. For the purposes of practical applications, it is sufficient to restrict attention to functions defined by a finite, although possibly very large, number of variables; with this restriction, the conventional definition of probability density applies. On the other hand, the existence of an established theory of integration on function spaces allows us to view numerical results as *approximations* to more exact results that correspond to the limit as the number of degrees of freedom tends to infinity.

The nature of the problem may be illustrated with reference to the one-dimensional probability density. Suppose that ξ is a one-dimensional random variable and $w(\xi)$ is the associated probability density. Then $w(\xi)$ is a function such that

$$\int_{-\infty}^{\infty} w(\xi)\mathrm{d}\xi = 1 \tag{1}$$

and the probability of finding ξ in the interval $(\xi_1, \xi_1 + \delta\xi)$ is $w(\xi_1)\delta\xi$.

The average, or expected, value E of a function of ξ [say, $z(\xi)$] then takes the form

$$E\{z(\xi)\} = \int_{-\infty}^{\infty} z(\xi)w(\xi)\mathrm{d}\xi \tag{2}$$

The generalization of the preceding to an n-dimensional random variable is straightforward; we simply employ standard (e.g., Lebesgue) volume integrals. However, in the case in which ξ becomes a *function* [say, a fluctuating signal $\xi(t)$], the number of degrees of freedom has effectively become infinite and technical problems arise. These were first tackled in the early 1920s, as described, for example, in [2].

Suppose that the probability density $w(\xi)$ occurring in Eqs. (1) and (2) is replaced by a *probability functional* $W[\xi(t)]$. We then wish to replace, for example, Eq. (2) by an analogous equation of the form

$$E\{z[\xi(t)]\} = \int z[\xi(t)]W[\xi(t)]\mathrm{d}[\xi(t)] \tag{3}$$

By analogy with Eq. (1), we require $W[\xi(t)]$ to be normalized such that

$$\int W[\xi(t)]\mathrm{d}[\xi(t)] = 1 \tag{4}$$

Solutions to the problem of rigorously defining the *probability measure* $\mathrm{d}[\xi(t)]$ in Eqs. (3) and (4) are reviewed in [2].

A heuristic introduction to the probability functional $W[\xi(t)]$ of a general random process $\xi(t)$ is to be found in [3], in which it is defined to characterize the probability of a particular realization of the process. This probability is obtained by considering the combined distribution of the discrete set of random values $\xi(t_1), \ldots, \xi(t_N)$, for $(t_1 < \cdots < t_N)$, and compacting the points $t_1, \ldots, t_N$ such that $|t_{j+1} - t_j| \to 0$. As only sampled data will be considered in this paper, the limiting step is unnecessary, and for practical applications, all integrals may be interpreted in terms of finite-dimensional integration.

As a basis for subsequent applications, it is convenient to express the probability functional $W[\xi(t)]$ in the form

$$W[\xi(t)] = C\exp(-U[\xi(t)]) \tag{5}$$

where C is a normalization constant such that Eq. (4) is satisfied. $U[\xi(t)]$ will be referred to here as the *generalized energy function* (strictly also a functional). In applications to statistical physics, it is generally defined in terms of interactions [4] between elementary subunits, which, in a turbulence model, would be elementary increments in turbulence velocity. In a Gaussian process the interactions are of second order only. In principle, non-Gaussian processes can be treated by introducing higher-order interactions. However, the application of this method, which would introduce higher-order interactions between elementary *microscopic* increments in turbulence velocity, proves to be impractical as a means of modeling severe turbulence for practical applications. Instead, in this paper, strongly non-Gaussian turbulence is modeled by representing *macroscopic* fluctuations in terms of their (mathematical) entropy.

It follows from Eq. (5) that samples of the process $\xi(t)$ corresponding to equal values of $U[\xi(t)]$ have equal probability of occurrence (strictly, with respect to the measure $d[\xi(t)]$). Furthermore, the maximization of the probability $W[\xi(t)]$, subject to some constraint, corresponds to the minimization of $U[\xi(t)]$.

In the case in which $\xi(t)$ is a realization of a stationary Gaussian white-noise process on an interval $(0, T)$, then $U[\xi(t)]$ takes the form [3]

$$U[\xi(t)] = \frac{1}{2K}\int_0^T \xi^2(t)\mathrm{d}t \tag{6}$$

where T is the length of the time interval. The integral in Eq. (6) is a particular case of the norm

$$\|\xi(t)\|^2 = \int_{-\infty}^{\infty} \{\xi(t)\}^2\mathrm{d}t \tag{7}$$

where $\xi(t)$ is defined on a finite interval. This norm can also be expressed in the Fourier domain:

$$\|\xi(t)\|^2 = \pi\int_0^{\infty} |\Xi(i\omega)|^2\mathrm{d}\omega \tag{8}$$

where $\Xi(i\omega)$ is the Fourier transform of $\xi(t)$.

This alternative expression of the norm is convenient when considering correlated random signals. Specifically, if the uncorrelated Gaussian white-noise signal $\xi(t)$ is passed through a linear filter with frequency-response function $H(i\omega)$ to obtain an output $\psi(t)$ for which the values at different instants are correlated, the Fourier transform of $\psi(t)$ is given by

$$\Psi(i\omega) = H(i\omega)\Xi(i\omega) \tag{9}$$

and thus

$$\|\xi(t)\|^2 = \pi\int_0^{\infty} \frac{|\Psi(i\omega)|^2}{|H(i\omega)|^2}\mathrm{d}\omega = \|\psi(t)\|_H^2 \tag{10}$$

where $\|\psi(t)\|_H^2$ is defined by Eq. (10) to be an H-dependent norm that reduces to the norm in Eq. (8) when H is the identity.

Because the PSD of $\psi(t)$ is given [5] by

$$\phi(\omega) = |H(i\omega)|^2 \tag{11}$$

it follows from Eq. (10) that

$$\|\psi(t)\|_H^2 = \pi\int_0^{\infty} \frac{|\Psi(i\omega)|^2}{\phi(\omega)}\mathrm{d}\omega \tag{12}$$

The probability functional of the correlated Gaussian noise process $\psi(t)$ with power spectral density $\phi(\omega)$ is thus obtained by identifying the function $\|\xi(t)\|^2$ with the *spectral energy function* for $\psi(t)$ defined by the norm $\|\psi(t)\|_H^2$ in Eq. (12). In a stationary Gaussian process having PSD $\phi(\omega)$, samples having equal spectral energy, given by Eq. (12), thus have equal probability of occurrence.

III. Wavelet Representation of Signal Structure

Whereas in a Gaussian process, the probability of an individual sample depends only upon its spectral energy, in non-Gaussian turbulence, the probability of a gust pattern will be seen to depend not only on its energy, but also upon the manner in which the energy is distributed among its component fluctuations. As a basis for quantifying the energy distribution within a given structured gust pattern, an economical numerical signal representation consists of the superposition of a finite number n of functions in the form of discrete *wavelets*, each of which is specified by parameters m_i, s_i, and T_i, which determine, respectively, its amplitude, location (in space or time), and scale. Specifically, this signal representation is of the following form:

$$g(t) = \sum_{i=1}^{n} m_i \bar{F}\left(\frac{t - s_i}{T_i}\right) \tag{13}$$

where, in the applications discussed in this paper, the function $\bar{F}(t)$, referred to as the *mother wavelet*, is taken to be a positive pulse, typically of the form $\exp(-t^2)$. Previous numerical applications of such positive wavelets have been described in [6,7].

In the terminology of the previous section, the function $g(t)$ given by Eq. (13) is taken not to represent turbulence directly, but rather the signal $\xi(t)$, which is used as an input to the linear shaping filter $H(i\omega)$ [Eq. (9)], for which the output represents the turbulence-velocity component $\psi(t)$. For example, if $H(i\omega)$ is taken to represent pure integration, then each term in the velocity component $\psi(t)$ resulting from Eq. (13) takes the form of a smooth incremental ramp, frequently taken to represent a discrete gust.

The design-envelope requirement for limit loads encountered during flight in continuous turbulence is prescribed in the airworthiness regulations [1] in terms of a turbulence-velocity component $\psi(t)$ having a specified power spectral density $\phi(\omega)$. For the vertical component of turbulence, the prescribed power spectral density takes the von Kármán form [8]:

$$\phi(\omega) = \frac{L}{\pi V}\frac{1 + 8/3(T\omega)^2}{(1 + (T\omega)^2)^{11/6}}, \qquad T = 1.339L/V \tag{14}$$

where L is the scale length, specified to be 750 m, and V is the true airspeed of the aircraft.

Corresponding to this spectral density, the *causal* linear filter having frequency-response function $H(i\omega)$ defined by Eq. (11) is given by [9]

$$H(i\omega) = \sqrt{\frac{L}{\pi V}}\frac{1 + \sqrt{8/3}Ti\omega}{(1 + Ti\omega)^{11/6}} \tag{15}$$

When $H(i\omega)$ is given by Eq. (15), the resulting elementary turbulence components, each corresponding to a single term in Eq. (13), take the form [9] of a ramp followed by a washout. At small scales ($T_i \ll T$), these elementary gusts are dominated by the ramp, followed by a gradual decay, whereas for scales $T_i \approx T$, they become approximately pulse-shaped. For scales $T_i \ll T$, gusts of equal probability have ramp amplitudes proportional to $T_i^{1/3}$.

IV. Probability and Entropy in Non-Gaussian Turbulence

A. Probability of Gust Patterns in Non-Gaussian Turbulence

In Sec. III, it has been explained that samples of equal probability in Gaussian models are associated with equal magnitudes of the spectral energy [Eq. (12)]. As described in [10,11], the relative amplitudes of equiprobable gust patterns have been measured in severe turbulence, recorded at both high and low altitudes. In this section, we show that these experimental results imply that the non-Gaussian structure of severe turbulence requires that the association between equality of probability and equality of spectral energy be modified.

The data analysis in [10,11] makes use of correlation detection [12,13], in which signal patterns of prescribed shape are detected through the occurrence of local extreme values (maxima and minima) in the output fluctuations of a digital filter that is matched to the shape of the signal pattern. The signal patterns comprise linear combinations of ramp-gust components (smoothed increments), and the associated detectors comprise associated linear combinations of smoothed increment detectors, each tuned to an individual ramp. Following identification, the extreme values in filter output are subjected to statistical analysis.

As described in [10], the method of statistical analysis involves the introduction of a nondimensional measure of fluctuation amplitude: the *moment-order parameter*. It is supposed that fluctuations of amplitude X have a probability density function $f(X)$, for which the qth moment is given by $\int f(X)X^q \mathrm{d}X$. Then, letting X_q denote the amplitude at which $f(X)$ makes its greatest contribution to the qth moment (i.e., at which the product $f(X)X^q$ is greatest), the resulting stationarity condition

$$X_q = -q\left\{\frac{f(X)}{f'(X)}\right\}_{Xq} \tag{16}$$

provides a means of labeling fluctuation amplitude in terms of the moment-order parameter q.

In [10], the parameter q is used to compare, over a range of amplitudes, the magnitudes of the extreme values in the responses of *pairs* of linear filters designed as detectors for gust patterns of differing complexity. Defining $p(n;q)$ to be the average ratio, at an amplitude specified by q, of the magnitudes of local extreme values in the output of a filter designed to detect an n-ramp pattern compared with those of a filter designed to detect a pattern comprising just a single ramp of similar scale, it is shown, by considering a range of q up to about 4 and values of $n \geq 2$, that in severe turbulence, $p(n;q)$ is a monotonically decreasing function of q.

This result contrasts with that obtained for a Gaussian process, for which it is demonstrated in [10] theoretically, and confirmed empirically, that $p(n;q)$ takes a value $p_G(n)$ independent of q. To a good approximation, which becomes exact in the particular case of a Gaussian process having PSD proportional to $(\text{frequency})^{-2}$, corresponding to the higher-frequency range of the Dryden spectrum sometimes used as a simplifying approximation in aeronautical applications [8]:

$$p_G(n) = n^{-1/2} \tag{17}$$

Taking $p_G(n)$ as a reference value, we may thus regard the ratio

$$r(n;q) = p(n;q)/p_G(n) \tag{18}$$

as an *amplitude-reduction factor* used to relate the amplitudes of extrema in the response of a pattern-detection filter, applied to a measured turbulence-velocity component, to the associated amplitudes that occur in a related Gaussian process.

As the magnitudes of local extrema in linear filter output are proportional to the amplitudes of the associated input signal patterns, the amplitude-reduction factors $r(n;q)$ may equally be applied to relate the relative amplitudes of gust patterns occurring in measured severe turbulence to the associated relative amplitudes of the same gust patterns occurring in the associated Gaussian process. Because it has already been shown, in Sec. II, that in a Gaussian process, the amplitudes of signal patterns having prescribed probability are determined by their spectral energy [Eq. (12)], it is convenient to replace the amplitude-reduction factors $r(n;q)$ by equivalent *energy-reduction factors*:

$$e(n;q) = \{r(n;q)\}^2 \tag{19}$$

Thus, in the case of the *non-Gaussian* turbulence-velocity component $\psi(t)$ having power spectral density $\phi(\omega)$, gust patterns comprising linear combinations of n ramp-gust components and having equal probability take equal values of the *reduced* spectral energy $e(n;q)\|\psi(t)\|_H^2$, where $e(n;q)$ is given by Eq. (19) and the norm $\|\psi(t)\|_H^2$ is given by Eq. (12).

Because it has been shown that, over a range of values of q up to about 4, in severe turbulence $p(n;q)$ is a monotonically decreasing function of q, it follows from Eqs. (18) and (19) that the same is true of $e(n;q)$. Values of q of particular significance are $q=2$ and 4. It is shown in [10] that at amplitudes corresponding to $q=2$ (second-order moments), the magnitudes of fluctuations in severe turbulence and in an associated surrogate Gaussian process having identical PSD are the same:

$$p(n;2) = p_G(n) \tag{20}$$

and thus

$$e(n;2) = 1 \tag{21}$$

It is also shown in [10] that measured values of $p(n;q)$ for $q \approx 4$, corresponding to fourth-order moments, are representative of the measured tails of the distributions. It is thus convenient to define $e(n)$ to be the value of $e(n;q)$ when q is approximately 4 and to take this value to be representative of the gusts of greatest amplitude that it is expected to encounter in severe turbulence.

Combining the analysis of measured severe-turbulence data in [10,11], the following empirical values have been obtained:

$$e(2) = 0.83, \qquad e(4) = 0.67, \qquad e(8) = 0.52 \tag{22}$$

Comparable results can be obtained from an analysis of data recorded during routine flying by civil aircraft, presented in [14]. These results were derived from gust encounters, during operational flying by civil aircraft, of sufficient intensity to produce a special event in which the aircraft response was at least 0.75 g. By curve-fitting to the results at the higher levels of intensity (in Fig. 23 of [14]), the following values for $e(n)$ have been derived:

$$e(2) = 0.77, \qquad e(4) = 0.55 \tag{23}$$

Gust patterns with more than four components have negligible effect on the associated large-magnitude measured loads.

The method of analysis applied in [14] was similar to that described in [10,11], involving the statistical comparison of the outputs of numerically simulated pairs of filters, in which the filters in each pair were tuned to input gust patterns of differing complexity (number n of discrete components).

It can be seen that the energy-reduction factors in Eqs. (23) are somewhat smaller than those in Eqs. (22), indicating greater departures from Gaussian statistics. Comparison with the full set of data in Fig. 23 of [14] shows that the former set of factors, which are from measured data obtained from severe-turbulence penetrations using specially instrumented research aircraft, are consistent with results in the middle range of intensities in the special-event data obtained from civil-aircraft severe turbulence encounters, obtained using more limited instrumentation. Indeed, the energy-reduction factors in Eqs. (22) are in very good agreement with those derived from one particular special event (event C1780) that lies in the middle range. However, the data in Fig. 23 of [14] suggest the overall conclusion that the energy-reduction factors in Eqs. (23) in fact correspond to the more severe events encountered by civil aircraft and are thus the more relevant to the prediction of design gust loads. The implementation of these energy-reduction factors in a model of severe turbulence is discussed in Sec. V.

B. Complexity and Entropy

It has been described, in Sec. IV.A, how for gust patterns in severe turbulence, the relationship between equal probability and equal spectral energy that holds for a Gaussian process is modified. As the gust intensity, as measured by q, increases, gust patterns comprising multiple elementary ramp components are reduced in amplitude relative to that of a gust of equal probability and comprising just a single ramp. As discussed in [10], this result is consistent with the hypothesis that the more severe gusts, associated with the tails of the distributions, tend to occur in short bursts. More general evidence for this phenomenon has been presented in [11].

The number n of elementary ramp components in a gust pattern may be interpreted as a simple measure of the degree of structure, or *complexity*, of the pattern. The energy-reduction factors $e(n;q)$ introduced in Sec. IV.A thus provide a quantitative measure of the influence of complexity on the associated probability. In other contexts, the concept of *entropy* has been widely employed as a measure of complexity, with increasing complexity being associated with increasing entropy. In particular, the concept of *information entropy* was introduced by Shannon [15,16] in the context of the mathematical theory of communication and subsequently developed by Jaynes [17] into a general principle of statistical inference, now known as the MAXENT (maximum-entropy) principle. As reviewed in [18], this principle tells us how to incorporate any statistical information about state variables of some system into a probability distribution $\{p_i\}$ over the possible states $\{i\}$ of the system. It tells us to choose that distribution that has the largest information entropy

$$S_I = -\sum_i p_i \log_2(p_i) \tag{24}$$

consistent with the given statistical information. In other words, maximize the entropy subject to the given statistical constraints.

In qualitative terms, a very sharply peaked distribution, with the probability distribution concentrated in a relatively small number of states, has a very low value of S_I, whereas if the distribution is spread over a large number of states, S_I is higher. If all of the probability is concentrated into one state, so that one of the p_i is 1 and all the others are 0, then S_I takes its smallest value (0). Conversely, the largest value of S_I arises when all of the p_i are equal.

In the application to discrete gust patterns, the p_i are identified with the absolute values of the wavelet coefficients m_i in Eq. (13), and the information entropy S_I [Eq. (24)] is taken to be a measure of the complexity of the associated gust pattern. A precedent for this use of the information entropy S_I [Eq. (24)], in which the p_i are identified with the amplitudes of components of a structured object, rather than directly with probabilities, arises in the context of image analysis. As described in [19], in maximum-entropy image restoration, or reconstruction, the p_i are identified with pixel intensities, or luminance elements, and S_I is incorporated as a regularizing term without which the effects of unmeasured spatial frequencies and noise would make the inversion problem of converting measured data to restored image ill-posed. This method is widely used in practice as a means of determining the least structured solution, consistent with the data, in which S_I imposes a constraint on the complexity of the structure.

In the present context, the information entropy S_I is used to replace n as a measure of the complexity of a discrete gust pattern, with entropy-based energy-reduction factors $e(S_I(q))$ and $e(S_I)$ replacing the energy-reduction factors $e(n;q)$ and $e(n)$, introduced in Sec. IV.A. Corresponding to the large-amplitude fluctuations in severe turbulence ($q \approx 4$), the following empirical expression for the energy-reduction factor has been derived by matching to measured data:

$$e(S_I) = (1 + aS_I^b)^{-1} \tag{25}$$

where S_I is given by Eq. (24), the p_i being identified with the absolute values of the wavelet coefficients m_i in Eq. (13), normalized to sum to unity. The energy-reduction factor $e(S_I)$ is a positive quantity, taking maximum value unity when the gust pattern comprises just a single ramp, and hence $S_I = 0$, and which decreases as the entropy of the gust pattern increases.

By matching to the values of $e(n)$ given by Eqs. (23), the following numerical values for the parameters in Eq. (25) have been obtained:

$$a = 0.3, \qquad b = 1.45 \tag{26}$$

Confirmation of the validity of this matching process is provided in Sec. V.B, which describes comparisons of linear aircraft response amplitudes derived using the $e(S_I)$ energy-reduction factors with the corresponding response amplitudes predicted by the SDG1 gust model, which uses $e(n)$. The application of Eq. (25) in the context of *nonlinear* aircraft response is discussed in Sec. V.C.

V. Applications

In this section, it is shown that the formulation of turbulence models in terms of energy and entropy can be applied in the process of predicting the associated aircraft loads/response. Section V.A is concerned primarily with Gaussian models of turbulence and linear models of aircraft response, as assumed in the current airworthiness requirements for aircraft loads. In Sec. V.B, the linearity of the model of response is retained, but the non-Gaussian structure of severe turbulence, described in Sec. IV, is introduced into the response-prediction process. Finally, Sec. V.C refers briefly to methods for predicting aircraft loads/response when the response equations are nonlinear for both Gaussian and non-Gaussian turbulence models.

A. Deterministic Spectral Procedure

In [20,21] it was demonstrated how the formulation of Gaussian turbulence models in terms of a spectral energy function could be applied to the prediction of system response. By introducing concepts related to matched-filter theory, an equivalence was demonstrated between deterministic and probabilistic design criteria for linear systems. It was shown that these results provided a basis for expressing system design requirements based on power spectral methods, usually implemented in the frequency plane [8], in terms of an equivalent worst-case analysis using deterministic inputs, in the time plane, subject to a constraint on the spectral energy [Eq. (12)]. Subsequent related work is described in [22–24].

In the standard implementation of the design-envelope requirement prescribed in the airworthiness requirements [1], a design load y_d is defined by the equation

$$y_d = \bar{A} U_\sigma \tag{27}$$

where U_σ is a turbulence intensity of prescribed magnitude; $\bar{A}$ is an aircraft-dependent dynamic response factor, calculated in the frequency plane as the ratio of the rms intensity of response to the rms intensity of a patch of turbulence having power spectral density $\phi(\omega)$ given by Eq. (14); and the equations representing the aircraft dynamics are required to be linear. The derivation of Eq. (27) also assumes [8] that the input comprises a sequence of independent Gaussian patches, each of sufficient duration that both input and response may be treated as stationary Gaussian processes.

Using results presented in [20,21], it was subsequently demonstrated in [24] that the design load y_d obtained using Eq. (27) can equivalently be calculated by a deterministic spectral procedure (DSP) in the time plane, as the maximum amplitude of aircraft response evaluated with respect to a deterministic family of gust inputs $\psi(t)$ subject to the inequality constraint

$$\|\psi(t)\|_H \leq U_\sigma \tag{28}$$

with the spectral energy $\|\psi(t)\|_H$ calculated according to Eq. (12).

It was further argued in [24] that, because this DSP for implementing the design-envelope requirement contains no reference to linearity, it may be regarded as applicable equally to linear and nonlinear aircraft responses. This conclusion is supported by the interpretation in Sec. II of $\|\psi(t)\|_H$ in terms of probability functionals, which demonstrates that samples having equal values of the spectral energy $\|\psi(t)\|_H$ have equal probability of occurrence in a Gaussian patch having spectral density $\phi(\omega)$ and hence, by summation, in a sequence of such Gaussian patches. The spectral-energy constraint in inequality (28) may thus be interpreted as a constraint on the probability of the input, quite independent of any properties of system response.

It is frequently convenient to model the response of an aircraft, which may be linear or nonlinear, to a turbulence input $\psi(t)$ by means of an equivalent *combined* system comprising the aircraft together with the linear prefilter or turbulence-shaping filter $H(i\omega)$ [Eq. (9)] subjected to a white-noise input $\xi(t)$. The deterministic spectral procedure, which requires the maximization of the aircraft response with respect to a family of gust inputs $\psi(t)$ subject to the inequality constraint (28), then takes an alternative form in which the response of the *combined* system is maximized with respect to a family of inputs $\xi(t)$ subject to the spectral-energy constraint

$$\|\xi(t)\| \leq U_\sigma \tag{29}$$

[using Eq. (10)].

When the aircraft response is linear, this problem has an analytical solution given by matched-filter theory. Specifically, the constrained input $\xi(t)$ that produces the maximum response then takes the form of the impulse-response function of the combined system reversed in time [21]. Passing this tuned input through the causal filter $H(i\omega)$ [Eq. (15)] then gives the associated tuned gust input $\psi(t)$. Details of this analytical solution, including the case of time-correlated multiple outputs, are given in [21]. The particular application to the calculation of time-correlated gust loads is illustrated in [22].

When the aircraft response is nonlinear, however, such analytical solutions are not applicable, and the DSP requires a numerical, rather than analytical, implementation, as will be outlined in Sec. V.C.

B. Statistical-Discrete-Gust Model

In this section, the linearity of the model of aircraft response is retained but the non-Gaussian structure of severe turbulence, described in Sec. IV, is introduced into the response-prediction process. It was primarily as a means of modeling the non-Gaussian characteristics of severe turbulence that the statistical-discrete-gust (SDG) model was developed. The SDG *method* [25,26] combines this model with an associated procedure for evaluating the statistics of aircraft response, involving a worst-case analysis to find the tuned gust pattern that maximizes the response subject to a constraint on the probability of the gust input. The statistical theory underlying the method involves an application of the Laplace approximation to an expression for the rate of occurrence of response peaks in the form of an integral over the space of input gust patterns, which demonstrates the dominant influence of the tuned gust pattern.

The version of the SDG model as it existed in 1989 is summarized in [26]. In this version, localized gust patterns were represented as clusters of discrete ramp-hold elements, with regions of constant wind velocity separating the ramps. Individual ramps in such a cluster were assumed to have equal probability (equipartition of probability). It was to this form of the model that the amplitude-reduction factors that quantify the effects of complexity on probability, described in Sec. IV.A, were first introduced to quantify one aspect of the non-Gaussian structure. A further feature of the model was the incorporation of a fractal (power law) representation [26] of the statistical relationship between the fluctuations of different size or scale. Fluctuations of equal probability are associated with equal values of an intensity parameter u_k, such that equiprobable patterns comprising just a single ramp, with velocity increment w and gradient distance H satisfy the equation $u_k = w/H^k$. To represent severe turbulence the scaling parameter takes [26] the value $k = 1/6$.

Subsequently, wavelet analysis was introduced as a basis for a reformulation of the SDG method in terms of the energy concepts introduced in Secs. II, III, and IV. New wavelet-based SDG algorithms for application to linear systems [11] were implemented in the scientific computer language MATLAB [9,27]. Two versions of the method were implemented: SDG2 and SDG1. The associated algorithms generate the worst-case responses when the SDG model is matched, respectively, to moderate turbulence and to measurements of large fluctuations in severe turbulence.

SDG2 may be interpreted as a discrete implementation of the DSP (Sec. V.A) in which elementary gust components are generated by passing pulse wavelets [Eq. (13)] of prescribed energy through the von Kármán shaping filter $H(i\omega)$ given by Eq. (15). For gradient distances significantly less than the scale length, this produces a family of ramp gusts satisfying approximately a one-third power law. It should be emphasized that the SDG2 gust model was never intended as a practical tool, but rather as a proof of concept in that it could be interpreted as an approximate implementation of the existing PSD turbulence model. This overlap between the SDG and PSD models was subsequently the subject of a NASA study [28].

In SDG1 the elementary gust components used to represent severe turbulence are generated by passing pulse wavelets [Eq. (13)] of prescribed energy through a modified von Kármán shaping filter [27], which, for gradient distances significantly less than the scale length, produces a family of ramp gusts satisfying approximately a one-sixth power law. SDG1 limits the maximum number of elementary ramp components in any gust pattern to four and also incorporates energy-reduction factors $e(n)$ to account for the reduced probability of meeting the more complex gust patterns in severe turbulence, as described in Sec. IV.A. In addition to the values for $e(2)$ and $e(4)$ given in Eqs. (23), obtained by matching to measured data, SDG1 also incorporates an interpolated value for $e(3)$, the full set of values being

$$e(2) = 0.77, \qquad e(3) = 0.635, \qquad e(4) = 0.55 \tag{30}$$

In this paper, as an implementation of the entropy concept introduced in Sec. IV.B, a new model, designated the SDG(E) model, is defined. This is identical to the SDG1 model except that, to take account of the effects of complexity upon probability in severe turbulence, the energy-reduction factors $e(n)$ given by Eqs. (30) are replaced by the entropy-based energy-reduction factor $e(S_I)$ [Eq. (25)], described in Sec. IV.B. Both SDG1 and SDG(E) may be interpreted as extensions of the DSP (Sec. V.A) to take account of the one-sixth scaling law characteristic of severe turbulence and in which the spectral-energy constraint [Eq. (29)], which is implemented in the DSP, is modified to take account of the reduced energy of the more complex gust patterns.

Comparisons have been made between the SDG1 and SDG(E) gust models and the PSD model of turbulence specified in the existing requirements for a variety of load quantities on mathematical models representative of an Airbus with underwing engines in two different configurations and flight conditions and with a turboprop aircraft. A full list of the response quantities evaluated, which include wing and tail bending moments and torques, is given in [11].

The maximum measured difference between the SDG1 and SDG (E) models was of order 1%. Two properties of the SDG models, which incorporate non-Gaussian properties of measured severe turbulence, influence their comparison with the PSD model, which is based on Gaussian statistics:

1) Compared with the PSD model, the ratio of the amplitudes of short gusts to the amplitudes of long gusts for a prescribed level of probability is increased in the SDG models on account of the difference between the respective scaling exponents: $k = 1/3$ in the PSD model and $k = 1/6$ in the SDG1 and SDG(E) models.

2) On account of the energy-reduction factors incorporated into the SDG models for a prescribed level of probability, the gust patterns comprising extended sequences of components are of lower amplitude than is the case with the PSD model.

A particular consequence that reflects both of the preceding properties is that the ratio of SDG design loads to PSD design loads tends to be greater for wing responses, such as wing-root bending loads, than for tail responses, such as rear fuselage and fin bending loads [11]. It follows, in particular, that if the SDG and PSD load criteria were to be matched for the case of wing response to vertical gusts, then an SDG-based criterion would tend to reduce the predicted tail loads in response to lateral gusts, with potential weight savings. Conversely, the optimization of safety margins for a given weight would result in increased wing strength. If a record were to be kept of all instances in which PSD gust design loads are exceeded during commercial flying, the SDG models of severe turbulence predict that the number of exceedances for wing loads will exceed that for (gust-induced) tail loads.

C. Nonlinear Aircraft Response

In Sec. V.A, an implementation of the DSP has been described that requires the maximization of the response of a combined system of aircraft plus shaping prefilter $H(i\omega)$ to a family of inputs $\xi(t)$ subject to the energy constraint given by Eq. (29). In Sec. V.B it was shown how the DSP may be modified, by energy-reduction factors that impose a constraint on the more complex gust patterns, to represent the non-Gaussian statistics of severe turbulence.

When the aircraft response is nonlinear, the DSP and its SDG-related modification require a numerical search for the maximum, or worst-case, response. For this purpose, it is convenient to represent the signal $\xi(t)$ as the discrete sum of pulse-wavelet components given by Eq. (13). Whether the search is based on a pure energy constraint, as in the DSP method, or the energy-plus-complexity constraint prescribed in the SDG model, response evaluation takes the form of a worst-case analysis to find the maximum magnitude of the response $y(H_n, U)$ to a family of equiprobable gust patterns of shape H_n, where n is the number of wavelet components in the gust pattern, and generalized energy $U[\xi(t)]$. For any prescribed value of U, $y(H_n, U)$ may be envisaged as a landscape over the space of gust patterns, for which the topography for a nonlinear system will in general depend upon U and the maximum magnitude of the response corresponds to the elevation of the highest hill.

In the application of the search to nonlinear systems, if the generalized energy U is varied in a continuous manner, the shape H_n of the tuned input pattern corresponding to the maximum peak in $y(H_n, U)$ may change slowly and smoothly with U (for a linear system, it is independent of U). Subsequently, however, a further small change in U may lead to a switch to a completely different tuned input pattern. Having established a new branch, the variation may again become smooth and gradual. Such a jump phenomenon will arise, for example, if there exist two distinct local maxima (hills) in $y(H_n, U)$ and a value of U at which the two maxima are of equal magnitude but growing at different rates with respect to U. The possibility of such discontinuities occurring when the system is nonlinear emphasizes the weakness of extrapolation and the importance of investigating system behavior at the appropriate value of probability, as specified by U.

A recommended search method based on the spectral-energy constraint of Eq. (29) as prescribed in the DSP, which has the capability to treat the problem of multiple maxima, is presented in [29]. This method employs a genetic search algorithm referred to as wavelet-based differential evolution. This combines the advantages of breeding and selection, as employed in standard genetic algorithms, with algebraic operations of addition, subtraction, and averaging. These allow the topography of the cost function $y(H_n, U)$ to be learned, reducing the danger of converging to a subsidiary, rather than the required global maximum value. In [29], the method is illustrated by application to an airbus-type aircraft with a nonlinear control system.

The preceding method is also described and illustrated in [30], which reviews the overall problem of meeting the current design-envelope requirement for aircraft loads in turbulence [1] when the response is nonlinear. Also illustrated in [30] is the modification of the preceding method in which the spectral-energy constraint in Eq. (29) is replaced by the combined energy-entropy constraint, introduced in Sec. IV.B, based on the SDG model of non-Gaussian severe turbulence. It was concluded in [30] that the implementation of this constraint in wavelet-based differential evolution offers not only a more realistic representation of the statistics of severe turbulence, but also has the computational advantage that, as a consequence of penalizing the more complex gust patterns, the resulting search for the worst-case gust input will tend to be concentrated into a more limited region of the search space. Other computational implementations of the SDG model to which this advantage of the energy/entropy constraint would also be applicable have been described in [31–33].

VI. Conclusions

1) It has been shown how the concept of a probability functional $W[\xi(t)]$ can be used to specify the probability of a random signal $\xi(t)$.

2) $W[\xi(t)]$ can be expressed in terms of a generalized energy function $U[\xi(t)]$, such that the maximization of the probability $W[\xi(t)]$ is associated with the minimization of $U[\xi(t)]$.

3) An associated time-plane method for evaluating the statistical response of a dynamic system involves finding the worst-case input, having prescribed generalized energy, and thus prescribed level of probability, that causes maximum response. Alternatively, in dual form, the worst-case input can be found as that which, for a prescribed amplitude of response, has maximum probability and thus minimum generalized energy.

4) The method for evaluating design loads in the existing airworthiness regulations, which assumes the dynamic response of the aircraft to be linear and the model of turbulence to be Gaussian, can be cast rigorously in the preceding form. In this case, the generalized energy takes the specific form of a *spectral energy* function, which depends only on the power spectral density of the turbulence.

5) The formulation of the statistical model of the turbulence in terms of the spectral energy function and the associated statistical evaluation of the response in terms of a worst-case input, jointly referred to as the DSP, is also applicable when the dynamic response is nonlinear.

6) When the non-Gaussian statistical structure of measured severe turbulence is taken into account, there is no longer a unique relationship between the probability of a sample and the spectral energy function. However, in this case, the spectral energy can be combined with the information entropy, which is a measure of the complexity of a fluctuation, to obtain a specification of the probability. It has been shown how the information entropy can conveniently be expressed in terms of a wavelet decomposition of the fluctuation.

7) It has been shown how, when the system response is linear, an energy-reduction factor based on the information entropy can be substituted for the energy-reduction factors previously used in the SDG1 gust model to quantify the probability of the more complex gust patterns in severe turbulence. With this substitution, the resulting gust model has been designated the SDG(E) model. Comparisons have been described between the predictions of gust loads on two aircraft types made by the SDG1 and SDG(E) models and the power-spectral-density model specified in the current requirements.

8) For application when the system response is nonlinear, reference has been made to a study indicating that the introduction of the information entropy not only leads to a more realistic measure of the probability of velocity fluctuations in severe turbulence, but also offers the computational advantage that the resulting search for the worst-case input will tend to be concentrated into a more limited region of the search space.

References

[1] "Part 25—Airworthiness Standards: Transport Category Airplanes," *Code of Federal Regulations*, Federal Aviation Administration and U.S. Dept. of Transportation, 1965.

[2] Jones, J. G., "Response of Linear Systems to Gaussian Process Input in Terms of Probability Functionals," Royal Aircraft Establishment, TM 86028, London, May 1986 http://www.stirling-dynamics.com/sdg_files/B-34.pdf [retrieved 13 May 2009].

[3] Stratonovich, R. L., "Optimum Nonlinear Systems for Isolating a Signal with Constant Parameters from Noise," *Nonlinear Transformations of Stochastic Processes*, edited by P. I. Kuznetsov, R. L. Stratonovich, and V. I. Tikhonov, Pergamon, New York, 1965, pp. 454–466.

[4] Ruelle, D., *Thermodynamic Formalism*, Addison–Wesley, Reading, MA, 1978.

[5] Papoulis, A., *Probability, Random Variables, and Stochastic Processes*, McGraw–Hill, New York, 1965.

[6] Watson, G. H., and Jones, J. G., "Positive Wavelet Representation of Fractal Signals and Images," Applications of Fractals and Chaos, edited by A. J. Crilly, R. A. Earnshaw, and H. Jones, Springer–Verlag, Berlin, 1993, pp. 117–135.

[7] Jones, J. G., Earwicker, P. G., and Foster, G. W., "Multiple-Scale Correlation Detection, Wavelet Transforms and Multifractal Turbulence," *Wavelets, Fractals and Fourier Transforms*, edited by M. Farge, J. C. R. Hunt, and J. C. Vassilicos, Clarendon, Oxford, 1993, pp. 235–250.

[8] Hoblit, F. M., *Gust Loads on Aircraft: Concepts and Applications*, AIAA Education Series, AIAA, Washington, D.C., 1988.

[9] Jones, J. G., and Watson, G. H., "Demonstration of Linear SDG Analysis," Stirling Dynamics, Ltd., Rept. SDL-231-TR-3, Bristol, England, U.K., June 1996 http://www.stirling-dynamics.com/sdg_files/E-06.pdf [retrieved 13 May 2009].

[10] Jones, J. G., "Measured Statistics of Multicomponent Gust Patterns in Atmospheric Turbulence," *Journal of Aircraft*, Vol. 44, No. 5, 2007, pp. 1559–1567.
doi:10.2514/1.27644

[11] Jones, J. G., "Documentation of the Linear Statistical Discrete Gust Method," U.S. Department of Transportation, Federal Aviation Administration, Office of Aviation Research, Rept. DOT/FAA/AR-04/20, July 2004.

[12] Jones, J. G., Foster, G. W., and Earwicker, P. G., "Wavelet Analysis of Gust Structure in Measured Atmospheric Turbulence Data," *Journal of Aircraft*, Vol. 30, No. 1, 1993, pp. 94–99.
doi:10.2514/3.46311

[13] Earwicker, P. G., and Jones, J. G., "Correlation Detection Using Multiple-Scale Filters and Self-Similar Noise Models," *Mathematics in Signal Processing 2*, edited by J. G. McWhirter, Clarendon, Oxford, 1990, pp. 337–346.

[14] Card, V., "Computer Simulation Studies of the Statistical Discrete Gust Method," Stirling Dynamics, Ltd., Rept. SDL-231-TR-4, Bristol, England, U.K., May 1996 http://www.stirling-dynamics.com/sdg_files/E-07.pdf [retrieved 13 May 2009].

[15] Shannon, C., "The Mathematical Theory of Communication," *Bell System Technical Journal*, Vol. 27, July 1948, pp. 379–423.

[16] Shannon, C., "The Mathematical Theory of Communication," *Bell System Technical Journal*, Vol. 27, Oct. 1948, pp. 623–656.

[17] Jaynes, E. T., "Information Theory and Statistical Mechanics," *Physical Review*, Vol. 106, No. 4, May 1957, pp. 620–630.
doi:10.1103/PhysRev.106.620

[18] Hestenes, D., "E. T. Jaynes: Papers on Probability, Statistics and Statistical Physics," *Foundations of Physics*, Vol. 14, No. 2, 1984, pp. 1–4.

[19] Narayan, R., and Nityananda, R., "Maximum Entropy Image Restoration in Astronomy," *Annual Review of Astronomy and Astrophysics*, Vol. 24, No. 1, 1986, pp. 127–170.
doi:10.1146/annurev.aa.24.090186.001015

[20] Jones, J. G., "An Equivalence Between Deterministic and Probabilistic Design Criteria for Linear Systems," *Journal of Sound and Vibration*, Vol. 125, No. 2, 1988, pp. 341–356.
doi:10.1016/0022-460X(88)90288-X

[21] Jones, J. G., "On the Implementation of Power-Spectral Procedures by the Method of Equivalent Deterministic Variables, Part 1, Analytical Background," Royal Aircraft Establishment, TM FS(F) 485, London, July 1982 http://www.stirling-dynamics.com/sdg_files/B-27.pdf [retrieved 13 May 2009].

[22] Pototzky, A. S., Zeiler, T. A., and Perry, B., III, "Calculating Time-Correlated Gust Loads Using Matched Filter and Random Process Theories," *Journal of Aircraft*, Vol. 28, No. 5, 1991, pp. 346–352.
doi:10.2514/3.46033

[23] Zeiler, T. A., "Matched Filter Concept and Maximum Gust Loads," *Journal of Aircraft*, Vol. 34, No. 1, 1997, pp. 101–108.
doi:10.2514/2.2141

[24] Jones, J. G., "Formulation of Design Envelope Criterion in terms of Deterministic Spectral Procedure," *Journal of Aircraft*, Vol. 30, No. 1, 1993, pp. 137–139.
doi:10.2514/3.46321

[25] Jones, J. G., "Statistical Discrete Gust Theory for Aircraft Loads. A Progress Report," Royal Aircraft Establishment, TR 73167, London, Nov. 1973 http://www.stirling-dynamics.com/sdg_files/B-11.pdf [retrieved 13 May 2009].

[26] Jones, J. G., "Statistical Discrete Gust Method for Predicting Aircraft Loads and Dynamic Response," *Journal of Aircraft*, Vol. 26, No. 4, 1989, pp. 382–392.
doi:10.2514/3.45771

[27] Jones, J. G., and Watson, G. H., "Algorithms for Linear SDG Analysis," Stirling Dynamics, Ltd., Rept. SDL-231-TR-2, Bristol, England, U.K., June 1996, http://www.stirling-dynamics.com/sdg_files/E-05.pdf [retrieved 13 May 2009].

[28] Perry, B., III, Pototzky, A. S., and Woods, J. A., "NASA Investigation of

a Claimed 'Overlap' Between Two Gust Response Analysis Methods," *Journal of Aircraft*, Vol. 27, No. 7, 1990, pp. 605–611. doi:10.2514/3.25329

[29] Watson, G. H., Gilholm, K., and Jones, J. G., "A Wavelet-Based Method for Finding Inputs of Given Energy Which Maximize the Outputs of Nonlinear Systems," *International Journal of Systems Science*, Vol. 30, No. 12, 1999, pp. 1297–1307. doi:10.1080/002077299291598

[30] Jones, J. G., "Nonlinear Aircraft Response in Severe Atmospheric Turbulence," *Journal of Aircraft* (to be published).

[31] Mehrotra, R., Karr, C. L., and Zeiler, T. A., "Genetic Algorithm for Optimizing the Gust Loads for Predicting Aircraft Loads and Dynamic Response," *Practical Applications of Computational Intelligence Techniques*, edited by L. Jain and P. De Wilde, Springer, New York, 2001, pp. 241–268.

[32] Karr, C. L., Zeiler, T. A., and Mehrotra, R., "Determining Worst-Case Gust Loads on Aircraft Structures Using an Evolutionary Algorithm," *Applied Intelligence: The International Journal of Artificial Intelligence, Neural Networks, and Complex Problem-Solving Technologies*, Vol. 20, No. 2, 2004, pp. 135–145. doi:10.1023/B:APIN.0000013336.08029.0c

[33] Steck, J. E., Tjondronegoro, M., and Rokhsaz, K., "Artificial Neural Networks for Maximum Gust Load Search: An Application in Statistical Discrete Methods," World Aviation Congress and Exposition, San Francisco, CA, SAE International, Paper 1999-01-5610, Oct. 1999.